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**Abstract**

We generalize the result of T. Komorowski and G. Papanicolaou published in [7]. We consider the solution of stochastic differential equation $d\mathbf{X}(t) = \mathbf{V}(t, \mathbf{X}(t))dt + \sqrt{2\kappa}d\mathbf{B}(t)$ where $\mathbf{B}(t)$ is a standard $d$-dimensional Brownian motion and $\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is a $d$-dimensional, incompressible, stationary, random Gaussian field decorrelating in finite time. We prove that the weak limit as $\varepsilon \downarrow 0$ of the family of rescaled processes $\mathbf{X}_\varepsilon(t) = \varepsilon\mathbf{X}(\frac{t}{\varepsilon^2})$ exists and may be identified as a certain Brownian motion.

## 1 Introduction

Consider the turbulent transport of a tracer particle modeled by the stochastic differential equation

$$\begin{cases} d\mathbf{X}(t;\omega,\sigma) = \mathbf{V}(t, \mathbf{X}(t);\omega)dt + \sqrt{2\kappa}d\mathbf{B}(t;\sigma), \\ \mathbf{X}(0;\omega,\sigma) = \mathbf{0}, \end{cases} \tag{1.1}$$

where $\mathbf{V}(t, \mathbf{x};\omega)$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is a $d$-dimensional random field over a certain probability space $\mathcal{T}_0 = (\Omega, \mathcal{V}, \mathbf{P})$ and $\mathbf{B}(t;\sigma)$, $t \geq 0$ is a standard $d$-dimensional Brownian motion over another probability space $\mathcal{T}_1 = (\Sigma, \mathcal{W}, \mathbf{Q})$. The constant $\kappa \geq 0$ stands for a molecular diffusivity of the medium. Let $\mathbf{E}$ and $\mathbf{M}$ denote the expectations in $\mathcal{T}_0$ and $\mathcal{T}_1$ respectively.

This model is widely used in physics literature to describe the motion in a turbulent flow. We are interested in a long time, large scale behavior of passive tracer over the product

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probability space. Namely, we consider the macroscopic scaling $\mathbf{x} \sim \mathbf{x}/\varepsilon$, $t \sim t/\varepsilon^2$. The rescaled process $\mathbf{X}_\varepsilon(\cdot)$ satisfies the following stochastic equation:

$$d\mathbf{X}_\varepsilon(t) = \frac{1}{\varepsilon}\mathbf{V}(\frac{t}{\varepsilon^2}, \frac{\mathbf{X}_\varepsilon(t)}{\varepsilon})dt + \sqrt{2\kappa}d\mathbf{B}(t).$$

We ask the natural question about the convergence in law of $\mathbf{X}_\varepsilon(t)$ as $\varepsilon \downarrow 0$.

This problem has a long history and several results, varying with the assumptions concerning the random velocity field $\mathbf{V}$, are well known. Let $\mathbf{V}$ be stationary, incompressible and centered. The first group of results deals with the velocity field having a so-called *stream matrix* $\mathbf{H}$, i.e.

$$\mathbf{V}(t, \mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{H}(t, \mathbf{x}).$$

Papanicolaou and Varadhan [10] and also Kozlov [8] proved the convergence of rescaled processes given by a time-independent field $\mathbf{V}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, with the stationary stream matrix $\mathbf{H}(\mathbf{x})$, under the assumption of boundedness of $\mathbf{V}$ and $\mathbf{H}$. The similar result was proven in [9] under the same conditions but for the time-dependent velocity field $\mathbf{V}(t, \mathbf{x})$. In [4] Fannjiang and Komorowski give the proof of convergence for the random fields $\mathbf{V}$ not bounded but with a finite $p$-th moment (for some $p > d + 2$).

In the second group we have results concerning the time-dependence assumptions imposed on the velocity field. Here we have for example the convergence for an Ornstein-Uhlenbeck velocity field of finite modes (Cramona and Xu [3]) and for a class of Markovian fields with strong mixing properties (Fannjiang and Komorowski [5]).

In this paper we follow the idea of Komorowski and Papanicolaou published in [7] which contributes to the last group of results. Assume that the field $\mathbf{V}$ is Gaussian, incompressible, stationary, centered and that it decorrelates in finite time. The authors considered the equation (1.1) with $\kappa = 0$ (without additional Brownian noise) and proved that the laws of the family of processes $\mathbf{X}_\varepsilon(t) = \varepsilon\mathbf{X}(\frac{t}{\varepsilon^2})$ converge as $\varepsilon \downarrow 0$ to that of the Brownian motion with covariance matrix given by

$$D_{ij} = \int_0^\infty \mathbf{E}[V_i(t, \mathbf{X}(t))V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t))V_i(0, \mathbf{0})]dt, \quad i, j = 1, \ldots, d.$$

We show that convergence still holds in the presence of molecular diffusivity $\kappa > 0$. Namely, we prove that the weak limit of the laws of $X_\varepsilon(\cdot)$ over $C([0, +\infty); \mathbb{R}^d)$ is a Brownian motion with covariance matrix

$$(1.2)\, D_{ij} = \int_0^{\infty} \mathbf{ME}[V_i(t, \mathbf{X}(t))V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t))V_i(0, \mathbf{0})]dt + 2\kappa\delta ij, \quad i, j = 1, \ldots, d.$$

This result confirms the turbulent diffusion hypothesis of G. I. Taylor coming from the early 1920-s (see [15]).

Let us describe the main steps of the proof. Thanks to the key assumption of the incompressibility of the field $\mathbf{V}$, in the proof we use the result of S. C. Port and C. Stone [11], namely the stationarity of the Lagrangian velocity process $(\mathbf{V}(t, \mathbf{X}(t)))_{t\geq 0}$. We develop, as in [7], the idea of the "transport operator" $Q$, being a linear operator acting on the space of elements integrable with respect to $\mathbf{P}$. $Q$ preserves densities and satisfies

$$\mathbf{ME}[V_i(s, \mathbf{X}(s))V_j(0, \mathbf{0})] = \mathbf{ME}\left[V_i(s - T, \mathbf{X}(s - T))Q[V_j(0, \mathbf{0})]\right],$$

for $s \geq T$, where $T$ is the decorrelation time of the field $\mathbf{V}$. Establishing estimates of the rate of convergence of the sequence $\{\|Q^n Y\|_{L^1}\}_{n\in\mathbb{N}}$ for any $Y$ measurable with respect to $\{V(t, \cdot), t \leq 0\}$ and satisfying $\mathbf{E}Y = 0$, we prove the convergence of the integrals in (1.2). In addition we show that for any $L > 0$

$$\mathbf{ME}[|\mathbf{X}_\varepsilon(t) - \mathbf{X}_\varepsilon(s)|^2 |\mathbf{X}_\varepsilon(u) - \mathbf{X}_\varepsilon(t)|] \leq C(u - s)^{1+\nu},$$

for any $0 \leq s \leq t \leq u \leq L$ and some constants $C, \nu > 0$. This gives us the tightness of the family $\{\mathbf{X}_\varepsilon(\cdot)\}$, $\varepsilon > 0$, in the Skorohod space $D([0, L], \mathbb{R}^d)$ for any $L > 0$ and by the continuity of trajectories also in $C([0, L], \mathbb{R}^d)$. Thanks to Stone [13], it gives tightness in $C([0, +\infty), \mathbb{R}^d)$. Finally we identify the limit as a certain Wiener measure with help of the Strook-Varadhan martingale problem.

This idea is strongly based on [7] and we make the references to lemmas presented there. We skip the proofs which may be generalized to our situation in a strightforward way. However, we present the complete argument for the results which are new or involve some major adjustments.

## 2 Notation and formulation of the main result

By $L^p := L^p(\Omega, \mathcal{V}, \mathbf{P})$ we understand the space of $L^p$-integrable random variables over the space $\mathcal{T}_0$ equipped with the standard $\|\cdot\|_p$ norm. Let $\mathbf{E}[\,\cdot\,|\mathcal{A}]$ denote the conditional expectation with respect to some sub $\sigma$-algebra $\mathcal{A} \subset \mathcal{V}$.

Let $\tau_{t,\mathbf{x}} : \Omega \to \Omega$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$, be a group of measure preserving transformations i.e. such that the map $(t, \mathbf{x}, \omega) \mapsto \tau_{t,\mathbf{x}}(\omega)$ is jointly $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{V}$ to $\mathcal{V}$ measurable, $\tau_{t,\mathbf{x}}\tau_{s,\mathbf{y}} = \tau_{t+s,\mathbf{x}+\mathbf{y}}$, $\tau_{t,\mathbf{x}}^{-1}(A) \in \mathcal{V}$ and $\mathbf{P}[\tau_{t,\mathbf{x}}^{-1}(A)] = \mathbf{P}[A]$ for all $(t, \mathbf{x}), (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^d$, $A \in \mathcal{V}$. Here $\mathcal{B}(\mathbb{R}^d)$ denotes the $\sigma$-algebra of Borel subsets of $\mathbb{R}^d$.

Suppose now that $\tilde{\mathbf{V}} : \Omega \to \mathbb{R}^d$ is a $d$-dimensional random vector such that the random field given by $\mathbf{V}(t, \mathbf{x}; \omega) := \tilde{\mathbf{V}}(\tau_{t,\mathbf{x}}(\omega))$ satisfies the following conditions:

V1) it is centered, i.e. $\mathbf{E}\tilde{\mathbf{V}} = \mathbf{0}$,

V2) it is Gaussian, i.e. all its finite dimensional distributions are Gaussian random vectors,

V3) it is of divergence free, i.e. $\operatorname{div}\mathbf{V}(t, \mathbf{x}) := \sum_{i=1}^d \partial_{x_i} V_i(t, \mathbf{x}) \equiv 0$,

V4) its correlation matrix $\mathbf{R}(t, \mathbf{x}) := [\mathbf{E}[V_i(t, \mathbf{x})V_j(0, \mathbf{0})]]_{i,j=1,\dots,d}$ satisfies

$$|\mathbf{R}(0, \mathbf{0}) - \mathbf{R}(t, \mathbf{x})| + \sum_{i,j=1}^d |\partial_{ij}^2 \mathbf{R}(0, \mathbf{0}) - \partial_{ij}^2 \mathbf{R}(t, \mathbf{x})| \le \frac{C}{|\ln \sqrt{t^2 + |x|^2}|^{1+\eta}},$$

for some constant $C > 0$ and all $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. According to [2], Theorem 3.4.1, this guaranties a version of $\mathbf{V}$ being jointly continuous in $(t, \mathbf{x})$ and of $C^1$ class in $\mathbf{x}$.

V5) the field $\mathbf{V}$ decorrelates in finite time, i.e. there exists $T > 0$ such that for all $|t| \ge T$ and $\mathbf{x} \in \mathbb{R}^d$ we have $\mathbf{R}(t, \mathbf{x}) = \mathbf{0}$.

**Remark.** For a Gaussian field $G(t)$, where $t \in \mathcal{T}$ is some abstract parameter, we define a $d$-ball as related to the pseudo-metric $d(t_1, t_2) = [\mathbf{E}|G(t_1) - G(t_2)|^2]^{\frac{1}{2}}$. Let $N(\varepsilon)$ denote the entropy number of the field $\mathbf{V}$, i.e. the minimal number of $d$-balls, corresponding to $\mathbf{V}$, with radius $\varepsilon > 0$ needed to cover $\mathbb{R} \times \mathbb{R}^d$. Thanks to the condition $V4)$ this number can be estimated by

$$N(\varepsilon) \le K_1 \exp\left(K_2(d+1)\varepsilon^{-\frac{2}{1+\eta}}\right), \tag{2.1}$$

for some constants $K_1, K_2 > 0$ independent of $\varepsilon$. According to [1], p. 121, (2.1) will allow us to use some of the Borell-Fernique-Talagrand type of tail estimates later on.

Let $\mathbf{B}(\cdot)$ denote the standard $d$-dimensional Brownian motion starting from $\mathbf{0}$, considered over a certain probability space $\mathcal{T}_1 = (\Sigma, \mathcal{W}, \mathbf{Q})$, with $\mathbf{M}$ the mathematical expectation corresponding to the probability measure $\mathbf{Q}$.

Let us consider a probability space $\mathcal{T}_0 \otimes \mathcal{T}_1 = (\Omega \times \Sigma, \mathcal{V} \otimes \mathcal{W}, \mathbf{P} \otimes \mathbf{Q})$ and a stochastic process $\mathbf{X}(\cdot)$ over this space, given by the following stochastic differential equation

$$\begin{cases} d\mathbf{X}(t;\omega,\sigma) = \mathbf{V}(t, \mathbf{X}(t;\omega,\sigma);\omega)dt + d\mathbf{B}(t;\sigma), \\ \mathbf{X}(0;\omega,\sigma) = \mathbf{0}. \end{cases} \tag{2.2}$$

For simplicity we suppose $\sqrt{2\kappa} = 1$ in (1.1) but the proof is still valid for any $\kappa > 0$. For $\varepsilon > 0$ define $\mathbf{X}_\varepsilon(t) := \varepsilon \mathbf{X}(\frac{t}{\varepsilon^2}), t \geq 0$. We will prove the following result:

**Theorem 1** *Suppose that the d-dimensional random field* $\mathbf{V}$ *satisfies the conditions* $V1) - V5)$ *listed above. Then the integrals*

$$D_{ij} = \int_0^\infty \mathbf{ME}[V_i(t, \mathbf{X}(t))V_j(0, \mathbf{0}) + V_j(t, \mathbf{X}(t))V_i(0, \mathbf{0})]dt + \delta_{i,j}, \quad i, j = 1, \ldots, d, \tag{2.3}$$

*converge. The laws of the processes* $\{\mathbf{X}_\varepsilon(t)\}_{t \geq 0}$ *induced on the space* $C([0, +\infty), \mathbb{R}^d)$ *converge weakly, as* $\varepsilon \downarrow 0$*, to the law of the Brownian motion with covariance matrix* $\mathbf{D} = [D_{ij}]$.

Some additional notation: for any $x \in \mathbb{R}$, $Ent(x)$ denotes the biggest integer smaller than or equal to $x$; for any $x, y \in \mathbb{R}$, $x \wedge y$ denotes $\min(x, y)$.

# 3 Auxiliary lemmas

Consider the processes $\mathbf{Y}(\cdot)$ over the probabitity space $\mathcal{T}_0 \otimes \mathcal{T}_1$ given by the equations

$$\begin{cases} d\mathbf{Y}^{0,\mathbf{x}}(t;\omega,\sigma) = \mathbf{V}(t, \mathbf{Y}^{0,\mathbf{x}}(t;\omega,\sigma);\omega)dt + d\mathbf{B}(t;\sigma), \\ \mathbf{Y}^{0,\mathbf{x}}(0;\omega,\sigma) = \mathbf{x}. \end{cases}$$

Thanks to the assumptions $V1) - V5)$ we can apply the following result of Port and Stone [11]. First of all, given $\mathbf{x} \in \mathbb{R}^d$, the equation above determines a unique process $\mathbf{Y}^{0,\mathbf{x}}$. Next we have:

**Lemma 1** *(cf. [11], Theorem 3, p. 501) For any* $t \geq 0$ *the random map* $\mathbf{x} \mapsto \mathbf{Y}^{0,\mathbf{x}}(t)$ *preserves measure on* $\mathbb{R}^d$ *and satisfies*

$$\mathbf{Y}^{0,\mathbf{x}}(t;\tau_{\mathbf{y}}(\omega),\sigma) = \mathbf{Y}^{0,\mathbf{x}+\mathbf{y}}(t;\omega,\sigma).$$

*For all* $\mathbf{x} \in \mathbb{R}^d$, $\sigma \in \Sigma$ *and* $t \geq 0$ *the random element* $\omega \mapsto \tau_{\mathbf{Y}^{0,\mathbf{x}}(t;\omega,\sigma)}(\omega)$ *has the law* $\mathbf{P}$ *on* $\Omega$

■

As a corollary we get the following lemma:

**Lemma 2** *Let $\tilde{U} : \Omega \times \Sigma \to \mathbb{R}$ be any random variable s.t. $\tilde{U} \in L^1(\mathcal{T}_0 \otimes \mathcal{T}_1)$ and define*

$$U(t, \mathbf{x}; \omega, \sigma) := \tilde{U}(\tau_{t, \mathbf{Y}^{0,\mathbf{x}}(t;\omega,\sigma)}(\omega), \sigma).$$

*For any $\mathbf{x} \in \mathbb{R}^d$ and $\tilde{U} \in L^1(\mathcal{T}_0 \otimes \mathcal{T}_1)$ the process $\{U(t, \mathbf{x})\}_{t \geq 0}$ is strictly stationary over $\mathcal{T}_0 \otimes \mathcal{T}_1$.*

**Proof.** Using the integral form of expressing $\mathbf{Y}^{0,\mathbf{x}}(t)$ it is easy to see that for any $t, h \geq 0$ we have

$$\mathbf{Y}^{0,\mathbf{x}}(t + h; \omega, \sigma) = \mathbf{Y}^{0,\mathbf{x}}(h; \omega, \sigma) + \mathbf{Y}^{0,\mathbf{0}}(t; \tau_{h, \mathbf{Y}^{0,\mathbf{x}}(h;\omega,\sigma)}(\omega), \sigma).$$

Moreover, thanks to Lemma 1 it follows that the measure $\mathbf{P} \otimes \mathbf{Q}$ is preserved by the transformation $\theta_h : (\omega, \sigma) \mapsto (\tau_{h, \mathbf{Y}^{0,\mathbf{x}}(h;\omega,\sigma)}(\omega), \sigma)$. We finish our proof essentially with the same argument as used to prove Lemma 1, p. 240, of [7] $\blacksquare$

Recall that random field $\mathbf{V}(t, \mathbf{x}; \omega) = (V_1(t, \mathbf{x}; \omega), \ldots, V_d(t, \mathbf{x}; \omega))$ is Gaussian. Let us introduce some notation. As in [7], let $L^2_{a,b}$ denote the closure in $L^2$-norm of linear span of $V_i(t, \mathbf{x})$, $t \in [a, b]$, $\mathbf{x} \in \mathbb{R}^d$, $i = 1, \ldots, d$. By $\mathcal{V}_{a,b}$ we denote the $\sigma$-algebra generated by all random vectors from $L^2_{a,b}$. Let $L^{2\perp}_{a,b} = L^2_{-\infty,+\infty} \ominus L^2_{a,b}$ be the orthogonal complement of $L^2_{a,b}$ in $L^2_{-\infty,+\infty}$ and $\mathcal{V}^{\perp}_{a,b}$ the $\sigma$-algebra similarly generated by elements of $L^{2\perp}_{a,b}$. According to [12], Theorems 10.1 and 10.2, p.181, the $\sigma$-algebras $\mathcal{V}_{a,b}$ and $\mathcal{V}^{\perp}_{a,b}$ are independent. Let $\mathbf{V}_{a,b}(t, \mathbf{x})$ be the orthogonal projection of $\mathbf{V}(t, \mathbf{x})$ onto $L^2_{a,b}$, i.e. each component of $\mathbf{V}_{a,b}(t, \mathbf{x})$ is the projection of the corresponding component of $\mathbf{V}(t, \mathbf{x})$. We define $\mathbf{V}^{a,b}(t, \mathbf{x}) := \mathbf{V}(t, \mathbf{x}) - \mathbf{V}_{a,b}(t, \mathbf{x})$. Of course $\mathbf{V}_{a,b}(t, \mathbf{x})$ is $\mathcal{V}_{a,b}$-measurable while $\mathbf{V}^{a,b}(t, \mathbf{x})$ is $\mathcal{V}^{\perp}_{a,b}$-measurable. We can also see that $\mathbf{V}_{a,b}(t, \mathbf{x})$ is jointly continuous and $C^1$-smooth in $\mathbf{x}$, $\mathbf{P}$-a.s. (see e.g. [2], Theorem 3.4.1). Finally let $\mathcal{T}_{a,b}$ denote the probability space $(\Omega \times \Omega, \mathcal{V}_{a,b} \otimes \mathcal{V}^{\perp}_{a,b}, \mathbf{P} \otimes \mathbf{P})$.

Over the space $\mathcal{T}_0 \otimes \mathcal{T}_0 = (\Omega \times \Omega, \mathcal{V} \otimes \mathcal{V}, \mathbf{P} \otimes \mathbf{P})$ we define the random field $\tilde{\mathbf{V}}$ by

$$\tilde{\mathbf{V}}_{a,b}(t, \mathbf{x}; \omega, \omega') = \mathbf{V}_{a,b}(t, \mathbf{x}; \omega) + \mathbf{V}^{a,b}(t, \mathbf{x}; \omega').$$

Let $c \in \mathbb{R}$ and consider now the process given by the following stochastic differential equation:

$$(3.1) \qquad \begin{cases} d\tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t; \omega, \omega', \sigma) = \tilde{\mathbf{V}}_{a,b}(t + c, \tilde{\mathbf{X}}^{b,\mathbf{x}}_c(t); \omega, \omega')dt + d\mathbf{B}(t + c; \sigma), \\ \tilde{\mathbf{X}}^{b,\mathbf{x}}_c(b) = \mathbf{x}. \end{cases}$$

If $a=-\infty$ and $b=0$ we shorten the notation by writing

$$\tilde{\mathbf{X}}(\cdot)=\tilde{\mathbf{X}}_0^{0,\mathbf{0}}(\cdot). \tag{3.2}$$

The following lemma concerning the conditional expectation of the process along its trajectory is the compilation of the adapted versions of Lemma 3, p. 241, and Lemma 4, p. 242, of [7] and may be proved with help of the same argument as presented there.

**Lemma 3** *Let $f\in L^2(\Omega,\mathcal{V}_{-\infty,+\infty},\mathbf{P})$ and $-\infty\le a\le b\le+\infty$. Then, there exists $\tilde f\in L^2(\mathcal{T}_{a,b})$ such that $f(\omega)=\tilde f(\omega,\omega)$ and both $f$ and $\tilde f$ have the same probability distributions. Let $c\in\mathbb{R}$. Then, for any $C\in\mathcal{V}_{a,b}^{\perp}$ we have:*

$$\mathbf{M}\mathbf{E}[f(\tau_{0,\mathbf{X}^{b,\mathbf{0}}(t;\tau_{c,\mathbf{0}}(\omega),\sigma)}(\omega))\mathbf{1}_C(\omega)|\mathcal{V}_{a,b}]=$$

$$\mathbf{M}\mathbf{E}_{\omega'}[\tilde f(\tau_{0,\tilde{\mathbf{X}}_c^{b,\mathbf{0}}(t;\omega,\omega',\sigma)}(\omega),\tau_{0,\tilde{\mathbf{X}}_c^{b,\mathbf{0}}(t;\omega,\omega',\sigma)}(\omega'))\mathbf{1}_C(\omega')],$$

*where $\mathbf{E}_{\omega'}$ denotes the expectation applied to the $\omega'$ variable only* ■

## 4 Transport operator $Q$ and its properties

Recall that $\tilde{\mathbf{X}}$ is given by (3.2). For a given $(\omega,\sigma)$ define a transformation $Z^t_{\omega,\sigma}:\Omega\to\Omega$ by

$$Z^t_{\omega,\sigma}(\omega')=\tau_{0,\tilde{\mathbf{X}}(t;\omega,\omega',\sigma)}(\omega').$$

Let $J^t(\cdot;\omega,\sigma)$ be the probability measure on $(\Omega,\mathcal{V}^{\perp}_{-\infty,0})$ given by

$$J^t(A;\omega,\sigma)=\mathbf{P}\left[(Z^t_{\omega,\sigma})^{-1}(A)\right].$$

Denote by $\mathcal{D}$ the family of densities $f\in L^1(\Omega,\mathcal{V}^{\perp}_{-\infty,0},\mathbf{P})$, $f\ge0$, $\int_\Omega f d\mathbf{P}=1$. For any $f\in\mathcal{D}$ we define a measure on $(\Omega,\mathcal{V}^{\perp}_{-\infty,0})$ by

$$[Qf][A]=\mathbf{M}\int_\Omega J^T(A;\omega,\sigma)f(\omega)\mathbf{P}(d\omega). \tag{4.1}$$

Let $\mathcal{V}^s_{0,T}$ denote the $\sigma$-algebra generated by the random vectors $\mathbf{V}^{-\infty,0}(t,\mathbf{x})$, $0\le t\le T$, $\mathbf{x}\in\mathbb{R}^d$, and let $\mathcal{V}^0_{-\infty,0}=\tau_{T,\mathbf{0}}(\mathcal{V}^s_{0,T})\subseteq\mathcal{V}_{-\infty,0}$. As in [7] we conclude that for any $f\in\mathcal{D}$, $[Qf]$ is absolutely continuous with respect to $\mathbf{P}$ and that its Radon-Nikodym derivative is $\mathcal{V}^s_{0,T}$-measurable. Moreover we have $[Q\mathbf{1}]=\mathbf{P}$.

Now we define the transport operator acting on $f \in \mathcal{D}$:

$$Qf = \frac{d[Qf]}{d\mathbf{P}} \circ \tau_{-T,\mathbf{0}}. \tag{4.2}$$

Equivalently to Lemma 7, p. 247, of [7] we get that $Q : L^1(\Omega, \mathcal{V}_{-\infty,0}, \mathbf{P}) \to L^1(\Omega, \mathcal{V}_{-\infty,0}, \mathbf{P})$ is a positive linear operator, preserving densities, $Q\mathbf{1} = \mathbf{1}$ and it extends to a contraction in every $L^p(\Omega, \mathcal{V}_{-\infty,0}, \mathbf{P})$ space, $1 \le p \le \infty$. Now we will prove the following lemma, which displays the usefulness of transport operator $Q$.

**Lemma 4** *Let $f, g \in L^1(\Omega)$ be random variables $\mathcal{V}^{-\infty,0}$ and $\mathcal{V}_{-\infty,0}$-measurable respectively. Let $t \ge T > 0$. Then, we have*

$$\mathbf{ME}[f(\tau_{t,\mathbf{X}(t;\omega,\sigma)}(\omega))g(\omega)] = \mathbf{ME}[f(\tau_{t-T,\mathbf{X}(t-T;\omega,\sigma)}(\omega))Qg(\omega)].$$

**Proof.** To shorten the notation, from now on $d\mathbf{P}$, $d\mathbf{Q}$, $d\mathbf{P}'$ and $d\mathbf{Q}'$ will stand for $\mathbf{P}(d\omega)$, $\mathbf{Q}(d\sigma)$, $\mathbf{P}(d\omega')$ and $\mathbf{Q}(d\sigma')$ respectively and we will omit the argument $\sigma$. If not indicated otherwise, all the integrals should be understood as the integrals over the whole space $\Omega \times \Sigma$. Using Lemma 3 we may write

$$\begin{aligned} \mathbf{ME}[f(\tau_{t,\mathbf{X}(t;\omega)}(\omega))g(\omega)] = \mathbf{ME}[\mathbf{ME}[f(\tau_{t,\mathbf{X}(t;\omega)}(\omega))|\mathcal{V}_{-\infty,0}]g(\omega)] = \\ \iint f(\tau_{t,\tilde{\mathbf{X}}(t;\omega,\omega')}(\omega'))g(\omega)d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q} = \\ \iint f(\tau_{t,\tilde{\mathbf{X}}^{T,\mathbf{0}}(t;\tau_{0,\tilde{\mathbf{X}}(T)}(\omega),\tau_{0,\tilde{\mathbf{X}}(T)}(\omega'))}(\tau_{0,\tilde{\mathbf{X}}(T)}(\omega')))g(\omega)d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q}. \end{aligned} \tag{4.3}$$

The last equality above comes from the fact that

$$\tilde{\mathbf{X}}(t;\omega,\omega') = \tilde{\mathbf{X}}(T;\omega,\omega') + \tilde{\mathbf{X}}^{T,\mathbf{0}}(t;\tau_{0,\tilde{\mathbf{X}}(T;\omega,\omega')}(\omega),\tau_{0,\tilde{\mathbf{X}}(T;\omega,\omega')}(\omega')).$$

Let $p^{\omega,\omega'}(s,\mathbf{x},t,\mathbf{y})$ denote the density of probability distribution of the process $\tilde{\mathbf{X}}^{s,\mathbf{x}}(t;\omega,\omega')$, $t \ge s$. If we introduce the notation $\mathbf{y} := \tilde{\mathbf{X}}(T;\omega,\omega')$ we may rewrite (4.3) in the form

$$\iiint_{\mathbb{R}^d} f(\tau_{t,\tilde{\mathbf{X}}^{T,\mathbf{0}}(t;\tau_{0,\mathbf{y}}(\omega),\tau_{0,\mathbf{y}}(\omega'))}(\tau_{0,\mathbf{y}}(\omega')))p^{\omega,\omega'}(0,0,T,\mathbf{y})g(\omega)d\mathbf{y}d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q}, \tag{4.4}$$

for $t \ge T$. Since $t \ge T$, where $T$ is the decorelation time of the field $\mathbf{V}$, from the definition of $\tilde{\mathbf{X}}$ we get that in fact $\tilde{\mathbf{X}}^{T,\mathbf{0}}(t;\tau_{0,\mathbf{y}}(\omega),\tau_{0,\mathbf{y}}(\omega')) = \mathbf{X}^{T,\mathbf{0}}(t;\tau_{0,\mathbf{y}}(\omega'))$. Hence (4.4) is equal to

$$\iiint_{\mathbb{R}^d} f(\tau_{t,\mathbf{X}^{T,\mathbf{0}}(t;\omega')}(\omega'))p^{\omega,\tau_{0,-\mathbf{y}}(\omega')}(0,0,T,\mathbf{y})g(\omega)d\mathbf{y}d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q} =$$

$$\iiint_{\mathbb{R}^d} f(\tau_{t-T,\mathbf{X}^{T,\mathbf{0}}(t;\tau_{-T,\mathbf{0}}(\omega'))}(\omega'))p^{\tau_{-T,-\mathbf{y}}(\omega),\tau_{-T,-\mathbf{y}}(\omega')}(0,0,T,\mathbf{y})g(\tau_{-T,-\mathbf{y}}(\omega))d\mathbf{y}d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q} =$$

$$\iiint_{\mathbb{R}^d} f(\tau_{t-T,\mathbf{X}(t-T;\omega')}(\omega'))p^{\omega,\omega'}(-T,-\mathbf{y},0,0)g(\tau_{-T,-\mathbf{y}}(\omega))d\mathbf{y}d\mathbf{P}'d\mathbf{Q}'d\mathbf{P}d\mathbf{Q}.$$

Define

$$Q'g(\omega) = \iint_{\mathbb{R}^d} p^{\omega,\omega'}(-T,-\mathbf{y},0,0)g(\tau_{-T,-\mathbf{y}}(\omega))d\mathbf{y}d\mathbf{P}'d\mathbf{Q}' \tag{4.5}$$

for any $\mathcal{V}_{-\infty,0}$-measurable $g$. It is easy to see that it is an equivalent way of expressing the transport operator $Q$ introduced before. We get the assertion of our lemma $\blacksquare$

As a corollary we have

**Lemma 5** *Let $p \geq 0$, $N, k \in \mathbb{N}$ and $s_{k+2} \geq s_{k+1} \geq \ldots \geq s_1 \geq NT$, $i = 1, \ldots, d$. Assume that $Y \in L^1(\Omega, \mathcal{V}_{-\infty,0}, \mathbf{P})$. Then*

$$\mathbf{ME}\left[\left|\int_{s_{k+1}}^{s_{k+2}} \mathbf{V}(\rho, \mathbf{X}(\rho))d\rho\right|^p \mathbf{V}(s_1, \mathbf{X}(s_1)) \ldots \mathbf{V}(s_k, \mathbf{X}(s_k))Y\right] =$$

$$\mathbf{ME}\left[\left|\int_{s_{k+1}-NT}^{s_{k+2}-NT} \mathbf{V}(\rho, \mathbf{X}(\rho))d\rho\right|^p \mathbf{V}(s_1-NT, \mathbf{X}(s_1-NT)) \ldots \mathbf{V}(s_k-NT, \mathbf{X}(s_k-NT))Q^N Y\right]$$

$\blacksquare$

# 5 Rate of convergence of $\{Q^n Y\}_{n\geq 0}$

In this section we will prove the following lemma which will constitute the main tool in establishing diverse estimates later on.

**Lemma 6** *Let $Y \in L_2(\Omega, \mathcal{V}_{-\infty,0}, \mathbf{P})$ be a random variable such that $\mathbf{E}Y = 0$. Then for any $s > 0$ there exists a constant $C$ depending only on $s$ and $\|Y\|_{L^2}$ such that*

$$\|Q^n Y\|_{L_1} \leq \frac{C}{n^s}, \quad \textit{for all} \quad n \in \mathbb{N}.$$

**Proof.** This proof is a modification of the proof of Lemma 10, p. 249, of [7]. According to [6], the part of $J^T(d\omega'; \omega, \sigma)$ which is absolutely continuous with respect to $\mathbf{P}$, has a density given by the formula

$$\int_{\mathbb{R}^d} \frac{\nu_0^T(d\mathbf{x}; \omega, \omega', \sigma)}{G_T(0; \omega, \tau_{0,\mathbf{x}}(\omega'), \sigma)}, \tag{5.1}$$

where $\nu_0^T(U;\omega,\omega',\sigma)$ stands for the cardinality of those $\mathbf{y}\in U$ for which $\psi_t(\mathbf{y};\omega,\omega',\sigma)=0$ and

$$G_T(\mathbf{x};\omega,\omega',\sigma)=\det\nabla_{\mathbf{x}}\psi_t(\mathbf{x};\omega,\omega',\sigma),$$

where

$$\psi_t(\mathbf{x};\omega,\omega',\sigma)=\mathbf{x}+\tilde{\mathbf{X}}(t;\omega,\tau_{0,\mathbf{x}}(\omega'),\sigma). \tag{5.2}$$

Similarly as in [7], using the integral form of expressing $\tilde{\mathbf{X}}$ (coming from (3.1)) we get that

$$\frac{d}{dt}\nabla_{\mathbf{x}}\psi_t(\mathbf{0};\omega,\omega',\sigma)=\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\tilde{\mathbf{X}}(t;\omega,\omega',\sigma);\omega)[\nabla\psi_t(\mathbf{0};\omega,\omega',\sigma)-\mathbf{I}]+$$

$$+\nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}(t,\tilde{\mathbf{X}}(t;\omega,\omega',\sigma);\omega')\nabla\psi_t(\mathbf{0};\omega,\omega',\sigma)$$

and

$$\nabla_{\mathbf{x}}\psi_0(\mathbf{0};\omega,\omega',\sigma)=\mathbf{I}.$$

Let $\alpha(\rho)$ be a smooth function, increasing on $\rho\geq 0$, satisfying $\alpha(-\rho)=\alpha(\rho)$, $\alpha(0)=0$, and $\alpha(\rho)=\sqrt{\rho}$ for $\rho\geq 1$. Let $\varphi(\mathbf{x})=\alpha(|\mathbf{x}|)$ for $\mathbf{x}\in\mathbb{R}^d$. Fix $\gamma\in(\frac{1}{2},1)$ and for any $\lambda>0$ let us introduce a set

$$K_n(\lambda)=\left[\omega\in\Omega:\sup_{0\leq t\leq T}[|\mathbf{V}_{-\infty,0}(t,\mathbf{x})|+|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\mathbf{x})|]\leq\lambda(\varphi(\mathbf{x})+\log^\gamma n)\right].$$

According to the Remark of Section 2 and the Theorem 5.4, p. 121, of [2], we may find $\Lambda>0$ such that for $\lambda\geq\Lambda$ we will have

$$\mathbf{P}[\Omega\setminus K_n(\lambda)]=\mathbf{P}[K_n(\lambda)^c]\leq C\exp\left(C_1\lambda^{\frac{2}{2+\eta}}\right)\exp\left(-\frac{\lambda^2}{4\sigma_n^2}\right)\leq C\exp\left(-\frac{\lambda^2}{8\sigma_n^2}\right),$$

where

$$\sigma_n^2=\sup_{0\leq t\leq T,\mathbf{x}\in\mathbb{R}^d}\mathbf{E}\left[\frac{|\mathbf{V}_{-\infty,0}(t,\mathbf{x})|^2+|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\mathbf{x})|^2}{(\varphi(\mathbf{x})+\log^\gamma n)^2}\right]=$$

$$\frac{1}{\log^{2\gamma}n}\mathbf{E}\left[|\mathbf{V}_{-\infty,0}(t,\mathbf{0})|^2+|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\mathbf{0})|^2\right]=\frac{C}{\log^{2\gamma}n}.$$

Hence

$$\mathbf{P}[K_n^c]=\mathbf{P}[\Omega\setminus K_n(\lambda)]\leq C_{01}\exp\{-C_{02}\lambda^2\log^{2\gamma}n\}$$

for some constants $C_{01},C_{02}>0$. Take $\nu\in(0,1)$ such that $\frac{\Lambda}{\Lambda+1}e^{\nu(\Lambda+1)T}<1$ and $\nu\Lambda<1$. Define also two families of sets by

$$L_m=\left[\omega\in\Omega:\sup_{0\leq t\leq T}[|\mathbf{V}^{-\infty,0}(t,\mathbf{x})|+|\nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}(t,\mathbf{x})|]\leq\nu|\mathbf{x}|+\log^\gamma m\right],$$

$$S_p = \left[\sigma \in \Sigma : \sup_{0\le t\le T} |\mathbf{B}(t;\sigma)| \le \log^{2\gamma} p\right].$$

Using the fact that the field $\frac{\mathbf{V}(t,\mathbf{x})}{\sqrt{t^2+|\mathbf{x}|^2+1}}$ is $\mathbf{P}$-a.s. bounded, we get $\lim_{m\to+\infty} \mathbf{P}[L_m] = 1$. From the properties of Brownian motion we get

$$\mathbf{Q}[S_p^c] = \mathbf{Q}[\Sigma \setminus S_p] = \mathbf{Q}[\sup_{0\le t\le T} |\mathbf{B}(t;\sigma)| \ge \log^{2\gamma} p] \le \frac{C}{\log^{4\gamma} p},$$

for any $p \ge 2$ and some constant $C > 0$.

The properties of $\alpha$ imply that there exists a constant $C_\nu$ such that $\varphi(\mathbf{x}) \le \nu|\mathbf{x}| + C_\nu$. For $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$ we get

$$\left|\frac{d(\tilde{\mathbf{X}}(t) - \mathbf{B}(t))}{dt}\right| = \left|\mathbf{V}_{-\infty,0}(t,\mathbf{x};\omega) + \mathbf{V}^{-\infty,0}(t,\mathbf{x};\omega')\right| \le$$

$$\nu(\Lambda+1)|\tilde{\mathbf{X}}(t)| + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m \le$$

$$\nu(\Lambda+1)|\tilde{\mathbf{X}}(t) - \mathbf{B}(t)| + \nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda\log^\gamma n + \log^\gamma m.$$

Hence, by Gronwall inequality

$$\sup_{0\le t\le T} |\tilde{\mathbf{X}}(t) - \mathbf{B}(t)| \le \frac{e^{\nu(\Lambda+1)T} - 1}{\nu(\Lambda+1)} \left[\nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m\right]$$

and since $\sigma \in S_p$ we get

$$(5.3)\ \sup_{0\le t\le T} |\tilde{\mathbf{X}}(t)| \le \frac{e^{\nu(\Lambda+1)T} - 1}{\nu(\Lambda+1)} \left[\nu(\Lambda+1)\log^{2\gamma} p + \Lambda C_\nu + \Lambda \log^\gamma n + \log^\gamma m\right] + \log^{2\gamma} p.$$

Using the estimates from above we write

$$\frac{d}{dt}|\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| \le \left|\frac{d}{dt}\nabla_{\mathbf{x}}\psi_t(\mathbf{0})\right| \le$$

$$\le \left|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\tilde{\mathbf{X}}(t);\omega)\right| [|\nabla\psi_t(\mathbf{0})| + 1] + \left|\nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}(t,\tilde{\mathbf{X}}(t);\omega')\right| |\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| \le$$

$$\le \sup_{0\le t\le T}\left|\nabla_{\mathbf{x}}\mathbf{V}_{-\infty,0}(t,\tilde{\mathbf{X}}(t);\omega)\right| [|\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| + 1] + \sup_{0\le t\le T}\left|\nabla_{\mathbf{x}}\mathbf{V}^{-\infty,0}(t,\tilde{\mathbf{X}}(t);\omega')\right| |\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| \le$$

$$\le [\nu(\Lambda+1)\sup_{0\le t\le T}|\tilde{\mathbf{X}}(t)| + \Lambda C_\nu + \Lambda\log^\gamma n + \log^\gamma m]|\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| +$$

$$+[\nu(\Lambda+1)\sup_{0\le t\le T}|\tilde{\mathbf{X}}(t)| + \Lambda C_\nu + \Lambda\log^\gamma n] \le a|\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| + b,$$

where $a$ and $b$ come from the estimate (5.3) of $\sup_{0\le t\le T}|\tilde{\mathbf{X}}(t)|$. Again from the Gronwall inequality we get

$$\sup_{0\le t\le T}|\nabla_{\mathbf{x}}\psi_t(\mathbf{0})| \le \frac{(a+b)e^{aT} - b}{a} \le \frac{(a+b)e^{aT}}{a} \le 2e^{aT} =$$

$$= 2\exp\left[T\Big[\exp(\nu(\Lambda+1)T)\Big]\Big(\nu(\Lambda+1)\log^{2\gamma}p + \Lambda C_\nu + \Lambda\log^\gamma n + \log^\gamma m\Big)\right] \leq$$

$$\leq C_{21}e^{C_{22}\log^\gamma n}e^{C_{23}\log^\gamma m}e^{C_{24}\log^{2\gamma}p}$$

for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$.

Let $B(\mathbf{0}, R)$ denote the ball of radius $R$ centered in $\mathbf{0} \in \mathbb{R}^d$. We will now investigate the size of the set $[\mathbf{x} : \psi_T(\mathbf{x}) = \mathbf{0}]$ for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$. We have a following lemma which may be prooved in the same way as Lemma 11, p. 252 of [7].

**Lemma 7** *For $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$ the random set $[\mathbf{x} : \psi_T(\mathbf{x}) = \mathbf{0}]$ is nonempty and there exists a constant $C_{31} > 0$ independent of $n,\ m,\ p$ such that*

$$[\mathbf{x} : \psi_T(\mathbf{x}) = \mathbf{0}] \subseteq B(\mathbf{0}, C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p)) \quad \blacksquare$$

For $\omega' \in L_m$ and $\mathbf{x}_1 \in B(\mathbf{0}, C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p))$ we have

$$|\mathbf{V}^{-\infty,0}(t, \mathbf{x}; \tau_{0,\mathbf{x}_1}(\omega'))| = |\mathbf{V}^{-\infty,0}(t, \mathbf{x} + \mathbf{x}_1; \omega')| \leq$$

$$\leq \nu|\mathbf{x} + \mathbf{x}_1| + log^\gamma m \leq \nu|\mathbf{x}| + \nu C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p) + \log^\gamma m \leq$$

$$\leq \nu|\mathbf{x}| + (1 + \nu C_{31})(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p) \leq \nu|\mathbf{x}| + C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p),$$

provided $C_{31}$ chosen sufficently large, since $\nu \in (0, 1)$. Hence $\tau_{0,\mathbf{x}_1}(L_m) \subset L_{Ent(M)+1}$, where $\log^\gamma M = C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p)$.

$$\bigcup_{|\mathbf{x}| \leq C_{31}(\log^\gamma n + \log^\gamma m + \log^{2\gamma} p)} \tau_{0,\mathbf{x}}(L_m) \subseteq L_{Ent(M)+1}.$$

Using all these we have

$$\int_{\mathbb{R}^d} \frac{\nu_{\mathbf{0}}^T(d\mathbf{x}; \omega, \omega'\sigma)}{G_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'), \sigma)} \geq C_{41}\int_{\mathbb{R}^d} \frac{\nu_{\mathbf{0}}^T(d\mathbf{x}; \omega, \omega'\sigma)}{|\nabla_{\mathbf{x}}\psi_T(\mathbf{0}; \omega, \tau_{0,\mathbf{x}}(\omega'), \sigma)|^d} \geq$$

$$\geq C_{42}\frac{1}{e^{C_{23}\log^\gamma(Ent(M)+1)}}\frac{1}{e^{C_{22}\log^\gamma n}}\frac{1}{e^{C_{24}\log^{2\gamma} p}} \geq C_{43}\frac{1}{e^{C_{44}\log^\gamma n}}\frac{1}{e^{C_{45}\log^\gamma m}}\frac{1}{e^{C_{46}\log^{2\gamma} p}},$$

for $\omega \in K_n(\Lambda)$, $\omega' \in L_m$ and $\sigma \in S_p$. Hence

$$[Qf][A] = \int_\Omega\int_\Sigma J^T(A; \omega, \sigma)f(\omega)\mathbf{P}(d\omega)\mathbf{Q}(d\sigma) \geq$$

$$\int_A \Gamma(\omega')\mathbf{P}(d\omega')\int_\Omega\int_\Sigma f(\omega)\Delta(\omega, \sigma)\mathbf{P}(d\omega)\mathbf{Q}(d\sigma),$$

where

$$\Gamma(\omega') = C_{51}\frac{1}{e^{C_{45}\log^\gamma m}} \quad \text{for } \omega' \in L_{m+1}\setminus L_m,$$

$$\Delta(\omega,\sigma) = \frac{1}{e^{C_{44}\log^\gamma n}}\frac{1}{e^{C_{46}\log^{2\gamma} p}} \quad \text{for } \omega \in K_{n+1}\setminus K_n,\ \sigma \in S_{p+1}\setminus S_p.$$

Finaly we obtain the formula equivalent to the formula (50) of [7]:

$$Qf(\omega') \geq \Gamma(\tau_{-T,\mathbf{0}}(\omega'))\int_\Omega\int_\Sigma f(\omega)\Delta(\omega,\sigma)d\mathbf{P}d\mathbf{Q}.$$

Now denote $Y_n = Q^nY$. Choose the minimum of $\int_\Sigma\int_\Omega Y_n^+\Delta d\mathbf{P}d\mathbf{Q}$ and $\int_\Sigma\int_\Omega Y_n^-\Delta d\mathbf{P}d\mathbf{Q}$; say it is the first one. We have then

$$\|Y_{n+1}\|_{L^1} \leq \|QY_n^+\|_{L^1} - \|QY_n^-\|_{L^1} - \int_\Sigma\int_\Omega Y_n^+\Delta d\mathbf{P}d\mathbf{Q}\int_\Omega \Gamma d\mathbf{P} \leq$$

$$\|Y_n\|_{L^1} - \int_{S_n}\int_{K_n} Y_n^+\Delta d\mathbf{P}d\mathbf{Q}\int_\Omega \Gamma d\mathbf{P} \leq$$

$$\|Y_n\|_{L^1} - e^{-C_{44}\log^\gamma n}e^{-C_{46}\log^{2\gamma} n}\int_{S_n}\int_{K_n} Y_n^+ d\mathbf{P}d\mathbf{Q}\int_\Omega \Gamma d\mathbf{P} =$$

$$\|Y_n\|_{L^1} + e^{-C_{44}\log^\gamma n}e^{-C_{46}\log^{2\gamma} n}\int_\Omega \Gamma d\mathbf{P}\times$$

$$\times\left(\int_{S_n}\int_{K_n^c} Y_n^+ d\mathbf{P}d\mathbf{Q} + \int_{S_n^c}\int_{K_n^c} Y_n^+ d\mathbf{P}d\mathbf{Q} + \int_{S_n^c}\int_{K_n} Y_n^+ d\mathbf{P}d\mathbf{Q} - \int_\Sigma\int_\Omega Y_n^+ d\mathbf{P}d\mathbf{Q}\right).$$

Since $\int_\Omega Y_n d\mathbf{P} = 0$ we get $\int_\Omega Y_n^+ d\mathbf{P} = \frac{1}{2}\|Y_n\|_{L_1}$. From this and the Schwartz inequality we have

$$\int_{S_n}\int_{K_n^c} Y_n^+ d\mathbf{P}d\mathbf{Q} + \int_{S_n^c}\int_{K_n^c} Y_n^+ d\mathbf{P}d\mathbf{Q} = \int_{K_n^c} Y_n^+ d\mathbf{P} \leq \sqrt{\mathbf{P}[K_n^c]}\|Y_n^+\|_{L_2} \leq$$

$$\leq C_{61}e^{-C_{62}\log^{2\gamma} n}\|Y\|_{L_2},$$

$$\int_{S_n^c}\int_{K_n} Y_n^+ d\mathbf{P}d\mathbf{Q} = \mathbf{Q}[S_n^c]\int_{K_n} Y_n^+ d\mathbf{P} \leq \frac{C}{\log^{4\gamma} n}\sqrt{\mathbf{P}[K_n]}\|Y_n\|_{L_2} \leq \frac{C}{\log^{4\gamma} n}\|Y\|_{L_2},$$

for some positive constants. We conclude with

$$\|Y_{n+1}\|_{L^1} \leq \|Y_n\|_{L^1}\left(1 - C_{70}e^{-C_{72}\log^\gamma n}\right) + C_{71}e^{-C_{73}\log^{2\gamma} n},$$

where the constants depend only on $\|Y\|_{L_2}$. Following the same argument we get

$$\|Y_n\|_{L^1} \leq C_{71}\sum_{k=1}^n e^{-C_{73}\log^{2\gamma} k}\prod_{p=k+1}^n\left(1 - C_{70}e^{-C_{72}\log^\gamma p}\right) + \|Y\|_{L_1}\prod_{p=1}^n\left(1 - C_{70}e^{-C_{72}\log^\gamma p}\right).$$

Since $1-x \le e^{-x}$, the $k$-th term in the sum above may be estimated by:

$$C_{71} e^{-C_{73}\log^{2\gamma} k} \exp\left\{ -C_{70} \sum_{p=k+1}^{n} e^{-C_{72}\log^{\gamma} p} \right\}.$$

We will estimate the last expression in two separate cases. For $k > [\frac{n}{2}]$ we may estimate it by

$$\frac{C_{81}}{n^{C_{82}\log^{2\gamma-1}(\frac{n}{2})}}.$$

For $k < [\frac{n}{2}]$

$$\sum_{p=k+1}^{n} e^{-C_{72}\log^{\gamma} p} = \sum_{p=k+1}^{n} \frac{1}{p^{C_{72}/\log^{1-\gamma} p}} \ge n^{1-r},$$

where $r \in (0,1)$ and $n$ is sufficiently large. Hence the $k$-th term of the sum is estimated by $C_{91} e^{-C_{71} n^{1-r}}$. This is the end of the proof of our lemma ■

**Remark.** One can see that the proof above remains valid for the sequence of $L^p$-norms of the iterates $Q^n Y$ for any $p \in (1,2)$.

## 6 Tightness

To establish the tightness of the family $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, we will use the following lemma (see [7], Lemma 12, p. 259):

**Lemma 8** *Suppose that* $\{\mathbf{Y}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$ *is a family of processes with trajectories in* $C[0,\infty)$, *such that* $\mathbf{Y}_\varepsilon(0) = 0$ *for* $\varepsilon > 0$. *Suppose further that for any* $L > 0$ *there exist constants* $p, C, \nu > 0$ *such that for any* $0 \le s \le t \le u \le L$

$$\mathbf{ME}\left[|\mathbf{Y}_\varepsilon(t) - \mathbf{Y}_\varepsilon(s)|^2 |\mathbf{Y}_\varepsilon(u) - \mathbf{Y}_\varepsilon(t)|^p\right] \le C(u-s)^{1+\nu}. \tag{6.1}$$

*Then the family* $\{\mathbf{Y}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon > 0$, *is tight.* ■

We will show that the process $\{\mathbf{X}_\varepsilon(t)\}$ satisfies the condition (6.1) with $p=1$ and $\nu = \frac{1}{2}$. Let us introduce the notation

$$\mathbf{V}(\rho) := \mathbf{V}(\rho, \mathbf{X}(\rho;\omega,\sigma);\omega)$$

and similarly for each component $V_i$, $i = 1, ..., d$ of the field $\mathbf{V}$. To begin with, we show that

**Lemma 9** *For any $L > 0$ there exists a constant $C > 0$ such that for any $0 \le s \le t \le u \le L$, $\varepsilon > 0$ we have*

$$\mathbf{ME}\left[|\mathbf{X}_\varepsilon(t) - \mathbf{X}_\varepsilon(s)|^2\right] \le C(t-s). \tag{6.2}$$

**Proof.** The left hand side of the inequality above may be rewritten in the form

$$\mathbf{ME}\left[\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho + \varepsilon\left[\mathbf{B}(\frac{t}{\varepsilon^2};\sigma) - \mathbf{B}(\frac{s}{\varepsilon^2};\sigma)\right]\right|^2\right] \le \tag{6.3}$$

$$2\mathbf{ME}\left[\varepsilon^2\left|\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^2\right] + 2\mathbf{ME}\left[\left|\varepsilon\left[\mathbf{B}(\frac{t}{\varepsilon^2};\sigma) - \mathbf{B}(\frac{s}{\varepsilon^2};\sigma)\right]\right|^2\right].$$

For any $\varepsilon > 0$ the process $\{\varepsilon\mathbf{B}(\frac{t}{\varepsilon^2})\}_{t\ge0}$ has the law of the standard Brownian motion in $\mathbb{R}^d$, so

$$\mathbf{ME}\left[\left|\varepsilon\left[\mathbf{B}(\frac{t}{\varepsilon^2};\sigma) - \mathbf{B}(\frac{s}{\varepsilon^2};\sigma)\right]\right|^2\right] = \sum_{i=1}^d \mathbf{ME}\left[\left[\varepsilon\left[\mathbf{B}_i(\frac{t}{\varepsilon^2};\sigma) - \mathbf{B}_i(\frac{s}{\varepsilon^2};\sigma)\right]\right]^2\right] = d(t-s).$$

The first term of the right hand side of (6.3) may be estimated as follows. Using the stationarity of $\{\mathbf{V}(t)\}_{t\ge0}$ (see Lemma 2) we may write

$$\mathbf{ME}\left[\varepsilon^2\left|\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^2\right] = \varepsilon^2\sum_{i=1}^d\mathbf{ME}\left[2\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\int_{\frac{s}{\varepsilon^2}}^{\eta}V_i(\rho)V_i(\eta)d\rho d\eta\right] = \tag{6.4}$$

$$2\varepsilon^2\sum_{i=1}^d\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\left\{\sum_{k=1}^{Ent(\frac{\eta}{T})}\int_{\eta-kT}^{\eta-(k-1)T}\mathbf{ME}\left[V_i(\eta-\rho)V_i(0)\right]d\rho\right\} +$$

$$+2\varepsilon^2\sum_{i=1}^d\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\int_0^{\eta-Ent(\frac{\eta}{T})T}\mathbf{ME}\left[V_i(\eta-\rho)V_i(0)\right]d\rho.$$

Then, from the Schwartz inequality for the last term above we have:

$$2\varepsilon^2\sum_{i=1}^d\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\int_0^{\eta-Ent(\frac{\eta}{T})T}\mathbf{ME}\left[V_i(\eta-\rho)V_i(0)\right]d\rho \le \tag{6.5}$$

$$2\varepsilon^2\sum_{i=1}^d\int_0^{\frac{t-s}{\varepsilon^2}}T\|V_i\|_{L_2}^2 d\eta \le 2\varepsilon^2\sum_{i=1}^d T\|V_i\|_{L_2}^2\frac{t-s}{\varepsilon^2} = C(t-s),$$

for a constant $C > 0$. Now we will estimate the first sum of the right-hand side of (6.4). Notice that for $\rho \in [\eta - kT, \eta-(k-1)T]$ we have $\eta - \rho \in [(k-1)T, kT]$. From Lemma 5 we get

$$\mathbf{ME}\left[V_i(\eta-\rho)V_i(0)\right] = \mathbf{ME}\left[V_i(\eta-\rho-(k-1)T)Q^{k-1}V_i(0)\right].$$

Using the stationarity of $\{\mathbf{V}(t)\}_{t\geq 0}$ and the estimation coming from Lemma 6 we may majorize the k-th term of the sum under consideration by $\frac{C}{k^2}$, for some constant $C>0$ depending only on $\|\mathbf{V}\|_2^2$. Hence the whole sum will be bounded from above by $C'(t-s)$. This finishes the proof of (6.2) ■

Since the sum $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent, from the argument above we can deduce the following corollary:

**Corollary 1** *The integrals*

$$\int_0^{\infty}\mathbf{ME}\left[\mathbf{V}_i(\rho,\mathbf{X}(\rho))\mathbf{V}_j(0,\mathbf{0})\right]d\rho$$

*for $i,j=1,2,\ldots,d$ converge* ■

To make use of Lemma 8 we will now establish the estimates of:

$$\mathbf{ME}\left[|\mathbf{X}_\varepsilon(u)-\mathbf{X}_\varepsilon(t)|\,|\mathbf{X}_\varepsilon(t)-\mathbf{X}_\varepsilon(s)|^2\right]\leq \tag{6.6}$$

$$2\mathbf{ME}\left[\left|\varepsilon\int_{\frac{t}{\varepsilon^2}}^{\frac{u}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^2\right]+2\mathbf{ME}\left[\left|\varepsilon\int_{\frac{t}{\varepsilon^2}}^{\frac{u}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|\left|\varepsilon\left[\mathbf{B}(\frac{t}{\varepsilon^2})-\mathbf{B}(\frac{s}{\varepsilon^2})\right]\right|^2\right]+$$

$$+2\mathbf{ME}\left[\left|\varepsilon\left[\mathbf{B}(\frac{u}{\varepsilon^2})-\mathbf{B}(\frac{t}{\varepsilon^2})\right]\right|\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^2\right]+$$

$$+2\mathbf{ME}\left[\left|\varepsilon\left[\mathbf{B}(\frac{u}{\varepsilon^2})-\mathbf{B}(\frac{t}{\varepsilon^2})\right]\right|\left|\varepsilon\left[\mathbf{B}(\frac{t}{\varepsilon^2})-\mathbf{B}(\frac{s}{\varepsilon^2})\right]\right|^2\right]=I+II+III+IV.$$

We will estimate each of the four terms of the utmost right hand side of (6.6) separately. From (6.2) and the Schwartz inequality we get that both $II$ and $IV$ can be estimated from above by $C(u-s)^{\frac{3}{2}}$. Now turn to the $I$ term of (6.6). Define

$$\Gamma_{1,k}:=\left|\varepsilon\int_{\frac{t-s}{\varepsilon^2}-\rho-(k-1)T}^{\frac{u-s}{\varepsilon^2}-\rho-(k-1)T}\mathbf{V}(\tau)d\tau\right|,\quad \Gamma_{2,k}:=V_i(\eta-\rho-(k-1)T).$$

We transform $I$ in the similar way as we did in (6.4) to get

$$I\leq 4\varepsilon^2\sum_{i=1}^{d}\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\sum_{k=1}^{Ent(\frac{\eta}{T})}\int_{\eta-kT}^{\eta-(k-1)T}\mathbf{ME}\left[\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0))\right]d\rho+$$

$$+4\varepsilon^2\sum_{i=1}^{d}\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\int_0^{\eta-Ent(\frac{\eta}{T})T}\mathbf{ME}\left[\Gamma_{1,1}\Gamma_{2,1}V_i(0)\right]d\rho.$$

We will now estimate $\mathbf{ME}\left[\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0))\right]$. We have

$$\mathbf{ME}\left[\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0))\right] = \tag{6.7}$$

$$\int_{\Gamma_{1,k}\leq k^{\nu}(u-t)^{\frac{1}{2}}}\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0)) + \int_{\Gamma_{1,k}>k^{\nu}(u-t)^{\frac{1}{2}}}\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0)),$$

where the constant $\nu > 0$ will be determined later. The first integral above will be less or equal to $\frac{C}{k^2}(u-t)^{\frac{1}{2}}$. Since by (6.2) we have

$$\mathbf{P}\otimes\mathbf{Q}[\Gamma_{1,k} > k^{\nu}(u-t)^{\frac{1}{2}}] \leq \frac{1}{k^{2\nu}(u-t)}\|\Gamma_{1,k}\|^2_{L^2} \leq \frac{C}{k^{2\nu}},$$

we may write for the second term of (6.7)

$$\left|\int_{\Gamma_{1,k}>k^{\nu}(u-t)^{\frac{1}{2}}}\Gamma_{1,k}\Gamma_{2,k}Q^{k-1}(V_i(0))\right| \leq$$

$$(\mathbf{P}\otimes\mathbf{Q}[\Gamma_{1,k} > k^{\nu}(u-t)^{\frac{1}{2}}])^{\beta_1}(\mathbf{ME}\Gamma^2_{1,k})^{\frac{1}{2}}(\mathbf{ME}\Gamma^{\frac{1}{\beta_2}}_{2,k})^{\beta_2}\|Q^{k-1}(V_i(0))\|_{L^{\frac{1}{\beta_3}}} \leq \frac{C(u-t)^{\frac{1}{2}}}{k^2},$$

for some $\beta_1, \beta_2, \beta_3 > 0$, $\beta_1+\beta_2+\beta_3+\frac{1}{2} = 1$, $\nu\beta_1 > 0$. From the same argument it follows that

$$\int_0^{\eta-Ent(\frac{\eta}{T})T}\mathbf{ME}\left[\Gamma_{1,1}\Gamma_{2,1}V_i(0)\right] \leq \int_0^T C(u-t)^{\frac{1}{2}} \leq C'(u-t)^{\frac{1}{2}}.$$

Finally we have

$$I \leq 4\varepsilon^2\sum_{i=1}^d\int_0^{\frac{t-s}{\varepsilon^2}}d\eta\left(\sum_{k=1}^{Ent(\frac{\eta}{T})}\frac{C(u-t)^{\frac{1}{2}}}{k^2} + C'(u-t)^{\frac{1}{2}}\right) \leq C''(t-s)(u-t)^{\frac{1}{2}} \leq C''(u-s)^{\frac{3}{2}}.$$

Now, with the help of Hölder inequality , the proof of the estimation of the third term in (6.6)may be reduced to the problem of showing that

$$\left[\mathbf{ME}\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^3\right]^{\frac{2}{3}} \leq C(t-s).$$

Since

$$\mathbf{ME}\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho)d\rho\right|^3 = \varepsilon^2\mathbf{ME}\left[\left|\varepsilon\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\mathbf{V}(\rho_1)d\rho_1\right|\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}\int_{\frac{s}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}}V_i(\rho_2)V_i(\rho_3)d\rho_2 d\rho_3\right],$$

it suffices to show the inequality in the case of $\rho_1 \geq \rho_2 \geq \rho_3$ (other cases are also covered by the symmetry of the term under the last integral above). But now we can use the same method as presented in the proof of estimate of $I$ and we finaly get

$$III \leq C(u-s)^{\frac{3}{2}}.$$

This ends the proof of the inequality

$$\mathbf{ME}\left[|\mathbf{X}_\varepsilon(u)-\mathbf{X}_\varepsilon(t)|\,|\mathbf{X}_\varepsilon(t)-\mathbf{X}_\varepsilon(s)|^2\right]\le C(u-s)^{\frac{3}{2}}$$

and by the Lemma 8 the family of the laws of processes $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$, $\varepsilon>0$, is tight on $C([0,+\infty),\mathbb{R}^d)$.

# 7 Limit identification

To finish the proof of our theorem we have to show that there exists only one process that can be the weak limit of the family $\{\mathbf{X}_\varepsilon(t)\}_{t\ge 0}$ as $\varepsilon\downarrow 0$. We prove this with the help of Strook-Varadhan martingale identification theorem. We start with the following lemma:

**Lemma 10** *(cf. [7], Lemma 13, p. 260) For any $\gamma\in(0,1)$, $M\in\mathbb{N}$, $\psi:(\mathbb{R}^d)^M\to\mathbb{R}_+$ continuous and bounded, $0\le s_1\le\ldots\le s_M\le s$ and $i=1,\ldots,d$, there exists a constant $C>0$ such that for any $\varepsilon>0$ and $0\le s\le t\le L$*

$$|\mathbf{ME}[(X^i_\varepsilon(t+\varepsilon^\gamma)-Y^i_\varepsilon(t))\psi(\mathbf{X}_\varepsilon(s_1),\ldots,\mathbf{X}_\varepsilon(s_M))]|\le C\varepsilon.$$

**Proof.** We have

$$\begin{aligned}(7.1)\qquad &|\mathbf{ME}[(X^i_\varepsilon(t+\varepsilon^\gamma)-X^i_\varepsilon(t))\psi(\mathbf{X}_\varepsilon(s_1),\ldots,\mathbf{X}_\varepsilon(s_M))]| =\\ &\left|\mathbf{ME}\left[\left(\varepsilon\int_{\frac{t}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}+\varepsilon^{\gamma-2}}V_i(\rho)d\rho\right)\psi(\mathbf{X}_\varepsilon(s_1),\ldots,\mathbf{X}_\varepsilon(s_M))\right]\right|+\\ &\left|\mathbf{ME}\left[\varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right)-B_i\left(\frac{t}{\varepsilon^2}\right)\right)\psi(\mathbf{X}_\varepsilon(s_1),\ldots,\mathbf{X}_\varepsilon(s_M))\right]\right|.\end{aligned}$$

The second term above is equal to

$$\left|\mathbf{ME}\left[\varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right)-B_i\left(\frac{t}{\varepsilon^2}\right)\right)\right]\mathbf{ME}\left[\psi(\mathbf{X}_\varepsilon(s_1),\ldots,\mathbf{X}_\varepsilon(s_M))\right]\right|=0.$$

The first term of the right hand side of (7.1) may be rewritten in the form

$$\mathbf{ME}\left[\varepsilon\int_0^{\varepsilon^{\gamma-2}}V_i(\rho)\Psi d\rho\right],$$

where $\Psi$ is a $\mathcal{V}_{-\infty,0}$-measurable random variable for any $\sigma\in\Sigma$. We get

$$\mathbf{ME}\left[\varepsilon\int_0^{\varepsilon^{\gamma-2}}V_i(\rho)\Psi d\rho\right]\le\sum_{k=1}^{Ent(\frac{\varepsilon^{\gamma-2}}{T})}\varepsilon\int_{kT}^{(k+1)T}\mathbf{ME}[V_i(\rho)\Psi]d\rho+\varepsilon\int_0^T\mathbf{ME}[V_i(\rho)\Psi]d\rho.$$

The last integral above is of order $\varepsilon$. Since $\mathbf{ME}[V_i(\rho)] = 0$ we may estimate as we already did in the previous section:

$$\sum_{k=1}^{Ent(\frac{\varepsilon^{\gamma-2}}{T})} \varepsilon \int_{kT}^{(k+1)T} \mathbf{ME}[V_i(\rho)\Psi]d\rho = \sum_{k=1}^{Ent(\frac{\varepsilon^{\gamma-2}}{T})} \varepsilon \int_{kT}^{(k+1)T} \mathbf{ME}[V_i(\rho)(\Psi - \mathbf{ME}\Psi)]d\rho =$$

$$\sum_{k=1}^{Ent(\frac{\varepsilon^{\gamma-2}}{T})} \varepsilon \int_{kT}^{(k+1)T} \mathbf{ME}[V_i(\rho-(k-1)T)Q^{k-1}(\Psi - \mathbf{ME}\Psi)]d\rho \leq$$

$$\sum_{k=1}^{Ent(\frac{\varepsilon^{\gamma-2}}{T})} \varepsilon \int_{kT}^{(k+1)T} \frac{C}{k^2} d\rho \leq C'\varepsilon \blacksquare$$

Now we have

**Lemma 11** *Under the assumptions of Lemma 10, for any $\varepsilon > 0$ we have*

$$|\mathbf{ME}[[(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t))(X_\varepsilon^j(t+\varepsilon^\gamma) - X_\varepsilon^j(t)) - \varepsilon^\gamma D_{ij}]\psi(\mathbf{X}_\varepsilon(s_1), \ldots, \mathbf{X}_\varepsilon(s_M))]| = o(\varepsilon^\gamma),$$

*for $i, j = 1, \ldots, d$, where the matrix $[D_{ij}]$ is given by (2.3), i.e.*

$$D_{ij} = \int_0^\infty \mathbf{ME}[V_i(t)V_j(0) + V_j(t)V_i(0)]dt + \delta_{ij}.$$

**Proof.** Notice that

$$\mathbf{ME}[(X_\varepsilon^i(t+\varepsilon^\gamma) - X_\varepsilon^i(t))(X_\varepsilon^j(t+\varepsilon^\gamma) - X_\varepsilon^j(t))\psi(\mathbf{X}_\varepsilon(s_1), \ldots, \mathbf{X}_\varepsilon(s_M))] = \tag{7.2}$$

$$\mathbf{ME}\left[\varepsilon^2 \int_{\frac{t}{\varepsilon^2}}^{\frac{t+\varepsilon^\gamma}{\varepsilon^2}} \int_{\frac{t}{\varepsilon^2}}^{\frac{t+\varepsilon^\gamma}{\varepsilon^2}} V_i(\rho)V_j(\rho')d\rho d\rho' \psi\right] +$$

$$\mathbf{ME}\left[\varepsilon^2 \left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right)\left(B_j\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_j\left(\frac{t}{\varepsilon^2}\right)\right)\psi\right] +$$

$$\mathbf{ME}\left[\varepsilon^2 \int_{\frac{t}{\varepsilon^2}}^{\frac{t+\varepsilon^\gamma}{\varepsilon^2}} V_i(\rho)\left(B_j\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_j\left(\frac{t}{\varepsilon^2}\right)\right)d\rho\psi\right] +$$

$$\mathbf{ME}\left[\varepsilon^2 \int_{\frac{t}{\varepsilon^2}}^{\frac{t+\varepsilon^\gamma}{\varepsilon^2}} V_j(\rho)\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right)d\rho\psi\right] =$$

$$= I + II + III + IV.$$

It is easy to see that $|II - \varepsilon^\gamma \delta_{ij}\mathbf{ME}[\psi]| = 0$. Following [7] we also get the estimate:

$$|I - \varepsilon^\gamma \int_0^{+\infty} [V_i(t)V_j(0) + V_j(t)V_i(0)]dt\mathbf{ME}[\psi]| = o(\varepsilon^\gamma) \tag{7.3}$$

To finish the proof of the lemma we are left with the proof of the fact that the terms $III$ and $IV$ in (7.2) are of magnitude $o(\varepsilon^\gamma)$. Once again we may find a $\mathcal{V}_{-\infty,0}$-measurable random variable $\Psi$ such that $III$ may be rewritten in the form

$$\left|\varepsilon^2 \int_0^{\varepsilon^{\gamma-2}} \mathbf{ME}\left[V_i(\rho)B_j(\varepsilon^{\gamma-2})\Psi\right] d\rho\right|.$$

Notice that up to a term of order $o(\varepsilon^\gamma)$ we have

$$\varepsilon^2 \int_0^{\varepsilon^{\gamma-2}} |\mathbf{ME}\left[V_i(\rho)B_j(\varepsilon^{\gamma-2})\Psi\right]| d\rho =$$

$$\varepsilon^\gamma \frac{1}{Ent(\frac{\varepsilon^{\gamma-2}}{T})T} \int_0^{Ent(\frac{\varepsilon^{\gamma-2}}{T})T} |\mathbf{ME}\left[V_i(\rho)B_j(\varepsilon^{\gamma-2})\Psi\right]| d\rho.$$

Since $\varepsilon^{\gamma-2} \to +\infty$ as $\varepsilon \downarrow 0$ it suffices to show that

$$\lim_{N\uparrow+\infty} \frac{1}{N} \int_0^{NT} |\mathbf{ME}\left[V_i(\rho)B_j(NT)\Psi\right]| d\rho = 0.$$

We may write

$$\frac{1}{N} \int_0^{NT} |\mathbf{ME}\left[V_i(\rho)B_j(NT)\Psi\right]| d\rho = \tag{7.4}$$

$$\frac{1}{N} \int_0^{T} |\mathbf{ME}\left[V_i(\rho)B_j(NT)\Psi\right]| d\rho + \frac{1}{N} \int_T^{NT} |\mathbf{ME}\left[V_i(\rho)B_j(NT)\Psi\right]| d\rho =$$

$$\frac{1}{N} \int_0^{T} |\mathbf{ME}\left[V_i(\rho)B_j(NT)\Psi\right]| d\rho + \frac{1}{N} \int_0^{(N-1)T} |\mathbf{ME}\left[V_i(\rho)B_j((N-1)T)Q\Psi\right]| d\rho.$$

Continue this way and we get that (7.4) is equal

$$\frac{1}{N} \sum_{k=0}^{N-1} \int_0^{T} |\mathbf{ME}\left[V_i(\rho)B_j((N-k)T)Q^k\Psi\right]| d\rho. \tag{7.5}$$

Now let $\widetilde{\Psi} = \Psi - \mathbf{ME}\Psi$. Since $Q\mathbf{1} = \mathbf{1}$, (7.5) may be rewritten in the form

$$\frac{1}{N} \sum_{k=0}^{N-1} \int_0^{T} |\mathbf{ME}\left[V_i(\rho)B_j((N-k)T)Q^k\widetilde{\Psi}\right]| d\rho +$$

$$\frac{1}{N} \sum_{k=0}^{N-1} \int_0^{T} |\mathbf{ME}\left[V_i(\rho)B_j((N-k)T)\right] \mathbf{ME}[Q^k\widetilde{\Psi}]| d\rho.$$

But

$$\mathbf{ME}\left[V_i(\rho)B_j((N-k)T)\right] = \mathbf{M}[\mathbf{E}[V_i(\rho)]B_j((N-k)T)] = 0,$$

since $\mathbf{E}[V_i(\rho)] = 0$. The first term above may be estimated with help of Lemma 6. We have $\|Q^k\widetilde{\Psi}\|_{L^q} \le \frac{C}{k^2}$ for any $q \in [1,2)$, $k \in \mathbf{N}$, the constant $C > 0$ depending only on $q$ and $\|\widetilde{\Psi}\|_{L^q}$. And we estimate (7.5) from above by

$$\frac{1}{N}\sum_{k=0}^{N-1} T\|V_i\|_{L^{\frac{4q}{3q-4}}}\|B_j((N-k)T)\|_{L^4}\|Q^k\widetilde{\Psi}\|_{L^q} \le \frac{C'}{N}\sum_{k=0}^{N-1}\frac{\sqrt{N-k}}{k^2} \le \frac{C''}{\sqrt{N}} \underset{N\uparrow+\infty}{\longrightarrow} 0.$$

This ends the proof of Lemma 11 ■

**Lemma 12** *For any $\gamma \in (0,1)$ and $0 < \gamma' < \gamma$ there exists a constant $C > 0$ such that*

$$\mathbf{ME}[|\mathbf{X}_\varepsilon(t+\varepsilon^\gamma) - \mathbf{X}_\varepsilon(t)|^4] \le C\varepsilon^{2\gamma'}.$$

**Proof.** We have

$$\mathbf{X}_\varepsilon(t+\varepsilon^\gamma) - \mathbf{X}_\varepsilon(t) = \varepsilon\int_{\frac{t}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}+\varepsilon^{\gamma-2}} V_i(\rho)d\rho + \varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right).$$

Define

$$A = \varepsilon\int_{\frac{t}{\varepsilon^2}}^{\frac{t}{\varepsilon^2}+\varepsilon^{\gamma-2}} V_i(\rho)d\rho, \quad B = \varepsilon\left(B_i\left(\frac{t+\varepsilon^\gamma}{\varepsilon^2}\right) - B_i\left(\frac{t}{\varepsilon^2}\right)\right).$$

It suffices to find the estimates of the type *i)* $\mathbf{ME}[|A^4|] \le C_1\varepsilon^{2\gamma'}$, *ii)* $\mathbf{ME}[|A^3B|] \le C_2\varepsilon^{2\gamma'}$, *iii)* $\mathbf{ME}[|A^2B^2|] \le C_3\varepsilon^{2\gamma'}$, *iv)* $\mathbf{ME}[|AB^3|] \le C_4\varepsilon^{2\gamma'}$, *v)* $\mathbf{ME}[|B^4|] \le C_5\varepsilon^{2\gamma'}$ for some constants $C_1, \ldots, C_5 > 0$. It is easy to see that once we have established *i)* and *v)* we get *ii)* $-$ *iv)* by Schwartz inequality. But *v)* is obvious. To prove *i)* we have to find an appropriate estimate from above of the term

$$\varepsilon^4\int_0^{\varepsilon^{\gamma-2}} d\rho_1\int_0^{\rho_1} d\rho_2\int_0^{\rho_2} d\rho_3\int_0^{\rho_3}\mathbf{ME}[V_i(\rho_2)V_i(\rho_3)V_i(\rho_4)V_i(0)]d\rho_4, \tag{7.6}$$

and this is exactly the result of Lemma 15 of [7] ■

Now let $f \in C_0^\infty(\mathbb{R}^d)$. Let $\psi$ be as in Lemma 11 and let $t_m = s + m\varepsilon^\gamma$. From the Taylor expansion formula we have

$$\mathbf{ME}[(f(\mathbf{X}_\varepsilon(t)) - f(\mathbf{X}_\varepsilon(s)))\Psi] =$$

$$\sum_{m:s<t_m<t}\mathbf{ME}\left\{\langle[\mathbf{X}_\varepsilon(t_{m+1}) - \mathbf{X}_\varepsilon(t_m)], (\nabla f)(\mathbf{X}_\varepsilon(t_m))\rangle\Psi\right\} +$$

$$\frac{1}{2}\sum_{m:s<t_m<t}\mathbf{ME}\left\{\langle[\mathbf{X}_\varepsilon(t_{m+1}) - \mathbf{X}_\varepsilon(t_m)] \otimes [\mathbf{X}_\varepsilon(t_{m+1}) - \mathbf{X}_\varepsilon(t_m)], (\nabla\otimes\nabla f)(\mathbf{X}_\varepsilon(t_m))\rangle\Psi\right\} +$$

$$\frac{1}{6}\sum_{m:s<t_m<t}\mathbf{ME}\left\{\langle[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]\otimes[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]\otimes[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)],\right.$$
$$\left.(\nabla\otimes\nabla\otimes\nabla f)(\mathbf{X}_\varepsilon(\theta_m))\rangle\Psi\right\},$$

where $\theta_m$ is a point in the segment $[\mathbf{X}_\varepsilon(t_m),\mathbf{X}_\varepsilon(t_{m+1})]$. The first term of the sum above is estimated with help of Lemma 10 by the term of magnitude $o(\varepsilon^\gamma)$. The third one, by Hölder inequality and Lemma 12, may be estimated from above by $C\varepsilon^{\frac{3\gamma'}{2}}$. Choosing carefully $\gamma\in(0,1)$ and $\gamma'\in(0,\gamma)$ we get again the term of magnitude $o(\varepsilon^\gamma)$. The second term we may rewrite in the form

$$\frac{1}{2}\sum_{m:s<t_m<t}\mathbf{ME}\left\{\langle[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]\otimes[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]-D^\varepsilon,(\nabla\otimes\nabla f)(\mathbf{X}_\varepsilon(t_m))\rangle\Psi\right\}+$$
$$\frac{1}{2}\sum_{m:s<t_m<t}\mathbf{ME}\left\{\langle D^\varepsilon,(\nabla\otimes\nabla f)(\mathbf{X}_\varepsilon(t_m))\rangle\Psi\right\},$$

where

$$D^\varepsilon=\mathbf{ME}[[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]\otimes[\mathbf{X}_\varepsilon(t_{m+1})-\mathbf{X}_\varepsilon(t_m)]].$$

And this, by Lemma 11, up to the terms of order $o(\varepsilon^\gamma)$ is equal to:

$$\frac{1}{2}\sum_{m:s<t_m<t}\left[\sum_{i,j=1}^{d}\varepsilon^\gamma D_{ij}\mathbf{ME}[\partial^2_{ij}f(\mathbf{X}_\varepsilon(t_m))\Psi]+o(\varepsilon^\gamma)\right]. \tag{7.7}$$

Thus we get that the limit of the family of processes $\{\mathbf{X}_\varepsilon(t)\}_{t\geq 0}$ as $\varepsilon\downarrow 0$ must have the law $\mu$ on the space $C([0,+\infty),\mathbb{R}^d)$ such that for any $f\in C_0^\infty(\mathbb{R}^d)$,

$$f(\mathbf{x}(t))-\frac{1}{2}\sum_{i,j=1}^{d}D_{ij}\int_0^t\partial^2_{ij}f(\mathbf{x}(\rho))d\rho,\qquad t\geq 0$$

is a martingale under $\mu$. By the theorem of Strook and Varadhan (see e.g. [14]) it may be identified as a diffusion with the covariance matrix $\mathbf{D}=[D_{ij}]$. This is the end of the proof of our theorem ■

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